

# A Twisted Tale of Cochains and Connections

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In honor of the 60-th birthday of Tornike Kadeishvili

## Abstract

Early in the history of higher homotopy algebra [Sta63], it was realized that Massey products are homotopy invariants in a special sense, but it was the work of Tornike Kadeishvili that showed they were but a shadow of an  $A_\infty$ -structure on the homology of a differential graded algebra. Here we relate his work to that of Victor Gugenheim [Gug82] and K.T. (Chester) Chen [Che73a]. This paper is a personal tribute to Tornike and the Georgian school of homotopy theory as well as to Gugenheim and Chen, who unfortunately are not with us to appreciate this convergence.

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# 1 Introduction

Early in the history of higher homotopy algebra [Sta63], it was realized that Massey products are homotopy invariants in a special sense, but it was the work of Tornike Kadeishvili that showed they are but a shadow of an  $A_\infty$ -structure on the homology of a differential graded algebra. Here we relate his work to that of Victor Gugenheim [Gug82] and K.T. (Chester) Chen [Che73a]. However, in light of the thorough technical analysis by Huebschmann elsewhere in this volume [Hue09a] and his earlier survey in honor of Berikashvili [Hue99], this will be a more personal tribute to Tornike

and the Georgian school of homotopy theory as well as to Gugenheim and Chen, who unfortunately are not with us to appreciate this convergence<sup>1</sup>. Essential to this discussion are the notions of *twisting element* and special cases: *twisting cochain* and *flat connection*. In the latter guise, there are new applications on the border with mathematical physics with which we conclude this tribute to Tornike.

## 2 Contact with the Georgian schools of category theory and algebraic topology

Kadeishvili met Huebschmann at a meeting in Oberwolfach in 1985, then he worked with Huebschmann as a Humboldt Scholar at Heidelberg in 1987/88. (This collaboration produced, among other results, [HK91].) Meanwhile, Huebschmann, Gugenheim and I were at the International Topology Conference in Baku in 1987 and met several of the Georgian category and homotopy theorists, including Kadeishvili. Even though we were able to meet in person in Baku, relations with the Soviet Union were still such that along the way to Baku I served as a courier between Borel at the IAS in Princeton and Margolis in Moscow.

Since then, personal contact has increased in both directions with Kadeishvili and others visiting the West and major meetings bringing non-Georgians to interact in Tbilisi. Unfortunately, I have been unable to return myself, but two of my former students, Tom Lada and Ron Umble, have been my representatives. It was a pleasure to have Tornike in attendance at my 70th birthday (and Murray Gerstenhaber's 80th) fest in Paris in 2007.

## 3 Berikashvili's twisting elements

In 1968, Berikashvili introduced the functor  $\mathcal{D}$  in terms of “twisting elements” in a differential graded algebra  $\mathcal{A}$ . Such elements  $\tau$  are homogeneous and satisfy the equation

$$d_{\mathcal{A}}\tau = \tau\tau,$$

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<sup>1</sup>Some of the memories and references here are my own, especially the personal ones, but I also owe a great deal to help from our Georgian colleagues and from Johannes Huebschmann, who also provided mathematical insights for earlier drafts.

where  $\tau\tau$  is the product in  $\mathcal{A}$  of  $\tau$  with itself.

If, instead, the algebra is a differential graded Lie algebra, the equation is

$$d_{\mathcal{A}}\tau = 1/2[\tau, \tau].$$

The element  $\tau$  is necessarily of degree  $\pm 1$  (equal to that of  $d_{\mathcal{A}}$ ) so  $[\tau, \tau]$  need not be zero. Signs depend on conventions and notation.

**Remark 1** *This equation has a long and honorable history in various guises. When the algebra is that of differential forms on a Lie group, it is called the Maurer-Cartan equation. In deformation theory, it is the integrability equation. In mathematical physics, especially in the Batalin-Vilkovisky formalism, it is known as the Master Equation. At present, the name Maurer-Cartan equation seems to have the upper hand.*

Berikashvili's functor  $\mathcal{D}$  assigns to the dga  $\mathcal{A}$  the set of equivalence classes of twisting elements, the equivalence relation nowadays being called *gauge equivalence*. His point of view is quite appropriate to applications to deformation theory. On the other hand, his twisting elements include traditional *flat* connection forms  $\omega$  on principal  $G$ -bundles:

$$G \rightarrow P \rightarrow M.$$

That is,  $\omega \in \Omega^*(P, \mathfrak{g})$ , which is a differential graded Lie algebra for  $\mathfrak{g}$ , the Lie algebra of  $G$ . Modern generalizations include those initiated by Chen as well as those with  $G$  generalized to *higher structure* analogs.

## 4 Connections

*Connections* have been well established for a long time in differential geometry, at least as far back as Elie Cartan, but, in the generality we build on, the notion was introduced by Ehresmann [Ehr51], though at times the word refers to equivalent but distinct concepts. In particular, the word is sometimes used as a synonym for *covariant derivative*.

### 4.1 Ehresmann's connections

**Definition 1** *An Ehresmann connection on a (locally trivial) smooth fiber bundle  $p : E \rightarrow B$  is a splitting of vector bundles for the induced*

morphism

$$TE \rightarrow B \times_B TB$$

of vector bundles.

**Remark 2** *In fact, Ehresmann's definition was for a submersion  $p : E \rightarrow B$  and he proved that, when the fibers are compact, such a splitting implies  $p$  is actually locally trivial.*

To such a choice of horizontal subspaces in  $TE$ , there corresponds a *connection form*  $\omega$ . Here we will be concerned mostly with a connection form on something like a principal bundle. Classically, for  $\mathfrak{g}$  the Lie algebra of a Lie group  $G$  and  $\pi : P \rightarrow X$  a principal  $G$ -bundle, a *principal connection form* on  $P$  is a  $\mathfrak{g}$ -valued 1-form  $\omega \in \Omega^1(P, \mathfrak{g})$  which satisfies two conditions:

1.  $\omega$  restricts to the classical Maurer-Cartan  $\mathfrak{g}$ -valued 1-form on each fiber.
2.  $\omega$  is equivariant with respect to the adjoint  $G$ -action on  $P$ .

Notice that a *flat* principal Ehresmann connection form is a twisting element in the sense of Berikashvili.

## 4.2 Cartan's connections

Henri Cartan observed [Car50] that this could be expressed in terms of a morphism of graded-commutative algebras on which there is the action of a Lie group (though only the action of the Lie algebra  $\mathfrak{g}$  is necessary).

**Definition 2** *A  $\mathfrak{g}$ -algebra is a dgca (differential graded commutative algebra)  $A$  such that:*

*For each  $x \in \mathfrak{g}$ , there is a derivation called 'infinitesimal transformation'  $\mathcal{L}(x)$  (today usually known as the Lie derivative) and a derivation called 'interior product' or 'contraction'  $\iota(x)$  satisfying the relations:*

1.  $\mathcal{L} \rightarrow \text{Der} A$  is an injective dg Lie morphism
2.  $\iota([x, y]) = \mathcal{L}(x)\iota(y) - \iota(y)\mathcal{L}(x)$
3.  $\mathcal{L}(x) = \iota(x)d + d\iota(x)$ .

These derivations are respectively of degree 0 and degree  $\pm 1$ , opposite to the degree of  $d$ .

These  $\mathfrak{g}$ -algebras are also known as *Leibniz Pairs* [FGV95].

The universal example of such a  $\mathfrak{g}$ -algebra is the Cartan-Chevalley-Eilenberg cochain algebra  $\text{CE}(\mathfrak{g})$  for Lie algebra cohomology:

$$\text{CE}(\mathfrak{g}) := \text{Hom}(\Lambda \mathfrak{g}, \mathbf{R})$$

with the differential induced by extending the dual of the bracket as a derivation.

**Remark 3** *The originators expressed this in terms of alternating multilinear functions on  $\mathfrak{g}$ , which remains the correct formulation for infinite dimensional Lie algebras, as opposed to the exterior algebra on the dual of  $\mathfrak{g}$ .*

A Cartan connection  $\Omega^\bullet(P) \xleftarrow{\omega} \text{CE}(\mathfrak{g})$  is then defined as respecting the operations  $i(x)$  and  $\mathcal{L}(x)$  for all  $x \in \mathfrak{g}$ , but not necessarily respecting  $d$ .

If  $\omega$  is a *flat* connection, it has curvature zero, that is equivalent to respecting  $d$ , hence satisfying the Maurer-Cartan equation. Thus it is an example of a *twisting element*.

Once we are in the dg (differential graded) world, we could just as well take  $\mathfrak{g}$  to be a differential graded Lie algebra, using a completed tensor product  $\hat{\otimes}$  where necessary. We can also work with differential graded associative algebras. It was K.T. Chen who did this first in 1973 [Che73a] and Kadeishvili independently in 1980 [Kad80].

### 4.3 Chen's connections

One of Chen's major contributions was a method for computing the real homology of the based loop space on a manifold in terms of the homology of the manifold. He effected this via his *iterated integrals*, initially in [Che73a] but evolving over several subsequent papers. In a very accessible survey [Che77], he uses the language of his *formal power series connections*.

**Definition 3** *Let  $X$  be a graded vector space with basis  $\{X_i\}$ . A formal power series connection on a differentiable space  $M$  with values*

in a vector space  $X$  is an element  $\omega \in \Omega^*(M)[[X]]$  of the type

$$\omega = \sum w_I X_I$$

where  $I$  denotes a multiindex  $i_1 \cdots i_r$  and  $X_I = X_{i_1} \cdots X_{i_r}$  and the coefficients  $w_I$  are forms of positive degree on  $M$ .

The algebra  $\Omega^*(M)[[X]]$  can also be written as  $\Omega^*(M) \hat{\otimes} TX$  where  $TX$  is the tensor algebra on  $X$ .

Chen, by suitably identifying his tensor product, saw that his condition for flatness becomes that of a twisting cochain, as he acknowledges in [Che77] Definition 3.2.1. In fact, such Chen connections with curvature zero are twisting elements in Berikashvili's sense, though probably due to restricted communication with the Soviet Union, Chen did not reference Berikashvili. Contact between the western and USSR groups grew gradually, thanks to the lifting of restrictions in the USSR under perestroika. Unfortunately, this came too late for Chen whose response to the Georgian school we would very much like to have seen.

To provide a multiplicative chain equivalence between his model and the chains on the based loop space  $\Omega X$ , Chen made use of his iterated integrals. Thus his approach provided an 'analytic' alternative to Adams cobar construction; one that was very useful in algebraic geometry [Hai02].

## 5 Twisting cochains

The earliest occurrence, to my knowledge, of the term *twisting cochain* is in the fundamental 1959 paper of E.H.Brown: Twisted tensor products I [Bro59]. (In the Séminaire Henri Cartan 1956-57, there is the term *fonction tordante*, but that is in the context of simplicial sets, then known as 'complete semi-simplicial complexes'.) Several related papers emphasized twisted tensor products and twisted differentials without mentioning twisting cochains, but it is the twisting cochains that are most closely related to connections.

**Definition 4** *Given a coaugmented differential graded coalgebra  $C$  (with coaugmentation  $\eta : R \rightarrow C$ ) and an augmented differential graded algebra  $A$  (with augmentation  $\varepsilon : A \rightarrow R$ , (both differentials*

being of degree  $-1$ ), a twisting cochain  $\tau : C \rightarrow A$  is a linear map of degree  $-1$  satisfying the conditions

$$d_A \tau + \tau d_C = \tau \smile \tau$$

$$\varepsilon \tau = 0 \quad \text{and} \quad \tau \eta = 0.$$

The cup-product  $\smile$  is defined in the module of linear maps  $C \rightarrow A$  by using the coproduct  $\Delta$  on  $C$  and the product  $m$  on  $A$ : Given two maps  $f, g : C \rightarrow A$ ,

$$f \smile g = m(f \otimes g) \Delta$$

Again, if we take  $\text{Hom}(C, A)$  with cup product as  $\mathcal{A}$ , then twisting cochains are twisting elements in Berikashvili's sense.

By 1960, Gugenheim had access to a preprint of Brown's paper and became interested in the idea of a twisting cochain and its relation to the description of a simplicial fibre bundle as a 'twisted cartesian product' in his work with Barratt and Moore [BGM59].

The fundamental role of twisting cochains in *differential homological algebra* was developed by J.C. Moore [Moo71]. In 1974, Moore, together with Husemoller and Stasheff, emphasized this role and applied it to a 'classical' problem in algebraic topology [HMS74]. Readers of that paper may well surmise who had primary responsibility for which part.

### 5.1 Chen's Theorem

In 1973, Chen [Che73b] proved a result which can be paraphrased as follows. Let  $\Omega M$  denote the based loop space on  $M$  and  $T(s^{-1}H_\bullet(M))$  denote the tensor algebra on the *desuspension* of the vector space  $H_\bullet(M)$ .

**Theorem 1** *For a simply connected manifold  $M$ , there is a twisting element  $\omega \in \Omega^\bullet(M) \hat{\otimes} T(s^{-1}H_\bullet(M))$  with respect to a derivation  $\partial$  on  $T(s^{-1}H_\bullet(M))$  for which there is a map*

$$\Theta : C_\bullet(\Omega M) \rightarrow (T(s^{-1}H_\bullet(M)), \partial)$$

*giving an isomorphism in homology.*



**Remark 4**  $\Omega^\bullet(M) \hat{\otimes} T(s^{-1}H_\bullet(M))$  can be written as  $\Omega^\bullet(M, T(s^{-1}H_\bullet(M)))$ .

Soon after, Gugenheim focused on the fact that twisting cochains and homotopies of twisting cochains are at the heart of Chen’s work. This interest culminated in 1982 [Gug82] where Gugenheim gave an algebraic version of Chen’s theorem on the homology of the loop space, not restricted to the smooth setting and differential forms nor even real coefficients. For this,  $C_\bullet(M)$  is replaced by a suitable differential graded coalgebra  $C$  and  $\Omega C$  denotes Adams’ cobar construction on  $C$ . Gugenheim constructed a multiplicative perturbation  $\partial$  of the cobar differential on  $\Omega H(C)$  and a map  $\Omega C \rightarrow \Omega_\partial H(C)$  which is a purely algebraic analog of the map  $\Theta$  given by Chen’s iterated integrals. Also in the early 80s, Huebschmann made extensive use of twisting cochains and homological perturbations, [Hue09b] and references therein. For the early history of homological perturbation theory (HPT), the review MR1103672 of [GLS91] by Ronnie Brown is excellent.

There were independent developments in the USSR by Berikashvili, Kadeishvili, Saneblidze and others.

## 5.2 Kadeishvili’s theorem

In 1980 and quite independently, Kadeishvili proved the corresponding very basic result for algebras and the bar construction, which is denoted  $B$  and generalized to apply to  $A_\infty$ -algebras where needed:

**Theorem 2** *If  $A$  is an augmented differential graded algebra with  $H_\bullet(A)$  free as a module over the ground ring, then  $H_\bullet(A)$  admits an  $A_\infty$ -structure such that there exists a map of dgcoalgebras  $BH(A) \rightarrow BA$  inducing an isomorphism in homology.*

This result is sometimes referred to as a ‘minimality theorem’, which, I think, has the wrong emphasis and point of view. It is the transfer of structure up to homotopy that to me is most important.

Apparently  $A_\infty$ -structures caught on faster in Moscow and especially Tbilisi than in the US, where Gugenheim’s version came to Stasheff’s attention. In 1986 [GS86] together they made the connection with  $A_\infty$ -structures. Considerable western work was thus inspired by Chen’s ideas, whereas Berikashvili and Kadeishvili led the way in the “east”.

Once  $A_\infty$ -structures appeared in this context, it was natural to consider  $A$  itself being an  $A_\infty$ -algebra; this is what Kadeishvili did in 1982 [Kad82]. He also developed further the relation between  $A_\infty$ -structures and Massey products, which was only implicit in my early work.

One of the characteristic features of Chen's connections and Gugenheim's twisting cochains is that they include as special cases twisted tensor products which are acyclic.

## 6 The Lie and $L_\infty$ versions and mathematical physics

We can also consider a twisting function  $\tau : C \rightarrow L$  from a dg coalgebra to a dg Lie algebra. As far as I know, this first occurred in the context of rational homotopy theory in Quillen's seminal paper [Qui69].

The main advantage of using  $Hom$  is the manifest naturality and the avoidance of finiteness conditions. Similarly the originators of Lie algebra cohomology got it right: using alternating multilinear functions on a Lie algebra  $\mathfrak{g}$  rather than the exterior (Grassmann) algebra on the dual of  $\mathfrak{g}$ . This works for infinite dimensional Lie algebras as well. Connection forms with values in a Lie algebra play key roles in math and physics, so generalizations to values in an  $L_\infty$ -algebra are natural.

Just as a Lie algebra or dg Lie algebra  $\mathfrak{g}$  can be characterized by a 'quadratic' differential on the graded symmetric coalgebra on the (de)suspension of  $\mathfrak{g}$ , so  $L_\infty$ -algebras can be characterized by removing the quadratic restriction. However, there was a considerable lag in introducing and developing  $L_\infty$ -structures until they were needed in algebraic deformation theory [SS77] and string field theory [Zwi93]. They were however implicit in Sullivan's models in rational homotopy theory [Sul77] in 1977. A lot was going on that year!

Chen deals with Lie algebras primarily in his studies of fundamental groups, but Hain [Hai83] adapts Chen's twisting cochains in the form of twisting elements in  $A \hat{\otimes} L$  where  $L$  is a dg Lie algebra (with  $d$  of degree -1) and  $A$  is a dg commutative algebra (with  $d$  of degree +1). As a student of Chen's and with specific computational examples in mind, this setting is natural for Hain. Generalization to  $L_\infty$ -valued connections is needed for applications to mathematical

physics.

### 6.1 Bundles with $L_\infty$ -structure

As higher category theory was developed, mimicing homotopy theory, Lie 2-algebras (also known as infinitesimal crossed modules) appeared [BC04] and were recognized as special (very small)  $L_\infty$ -algebras. This led naturally to the differential homological algebra version of classical differential geometry, in particular, generalized connections, curvature and ‘all that’. However, the driving force in this recent development was application suggested by mathematical physics: differential graded string theory and even ‘5brane theory’ [SSS09a].

The first example of “higher bundles with connection” occurred with the fundamental (super)string coupling to the Neveu-Schwarz (NS)  $B$ -field  $B_2$  [AN71]. This  $B$ -field is a connection on a 2-bundle and appears in an action functional  $\int_\Sigma B_2$  for the string worldsheet (surface)  $\Sigma$ . Here a 2-bundle [Bar04] means a bundle with fibres which are at least 2-vector spaces [BC04], that is, a differential graded vector space of the form  $V = V_0 \oplus V_1$  with  $d$  of degree 1. Similarly, a connection  $B_6$  on a 6-bundle appears in an action functional  $\int_{\Sigma_6} B_6$  for the fivebrane worldvolume  $\Sigma_6$ . This and related matters are explained in [Fre00] in the language of differential characters and in [SSS09b, SSS09a] in the language of higher bundles.

To stay at the level of dgcas, we make the definition below. First notice that the definition of a  $\mathfrak{g}$ -algebra above applies to  $L_\infty$ -algebras  $\mathfrak{g}$  except that, for  $x$  of degree  $k$ , the degree of  $\mathcal{L}(x)$  is  $-k$  and that of  $i(x)$  is  $-k - 1$ .

**Definition 5** *An algebra of differential forms on a principal  $\mathfrak{g}$ -bundle over a smooth space  $X$  is a  $\mathfrak{g}$ -algebra in the sense of H. Cartan (we denote it  $\Omega^\bullet(Y)$ ) with a monomorphism  $\pi : \Omega^\bullet(X) \rightarrow \Omega^\bullet(Y)$  such that  $i(x)\pi = 0 = \mathcal{L}(x)\pi$  for all  $x \in \mathfrak{g}$ .*

The formal definition of such a connection can be stated in the Henri Cartan form. The graded commutative algebra  $\text{CE}(\mathfrak{g})$  has the usual operations  $i(x)$  and  $\mathcal{L}(x)$ . As before, an  $L_\infty$ -Cartan connection  $\Omega^\bullet(Y) \xleftarrow{A} \text{CE}(\mathfrak{g})$  is then defined as a graded algebra map injective on the dual of  $\mathfrak{g}$ , respecting  $i(x)$  and  $\mathcal{L}(x)$  for all  $x \in \mathfrak{g}$ , but not necessarily respecting  $d$ .

In most of the applications to physics,  $\mathfrak{g}$  is non-zero in a very, very small number of degrees. These are related to connected covers of  $BO$  or  $BU$ . For example, a *spin*-structure on a smooth space  $X$  corresponds to a lifting of the classifying map of  $TX$  to the 2-connected cover, a *string*-structure corresponds to a lifting to the 4-connected cover and a *Fivebrane*-structure corresponds to a lifting to the 8-connected cover. The standard Chern-Weil approach using the  $L_\infty$  version of the Weil algebra then applies to determine characteristic classes of bundles with such structures [SSS09b].

## 6.2 Higher spin structures, closed string field theory and $L_\infty$ -algebra

Recognition that the mathematical structure of sh-Lie algebras (=  $L_\infty$ -algebra) was appearing in physics first occurred in my discussions at UNC with Burgers (visiting van Dam) and then with Zwiebach at the third GUT Workshop in 1982. In their study of *field dependent gauge symmetries* for field theories for higher spin particles [Bur85, BBvD85, BBvD86], Behrends, Burgers and van Dam discovered what turned out to be an  $L_\infty$ -structure. In conversations, Burgers and I found we had common formulas, if not a common language.

As a generalization of Lie algebras, sh Lie algebras (now more commonly known as  $L_\infty$ -algebras) appeared in physics as symmetries or gauge transformations, though they were not presented as such initially in the physics literature [Bur85, BBvD86, Zwi93]. They were recognized as such in closed string field theory when Zwiebach and I were together at the third GUT Workshop in Chapel Hill. The corresponding Lagrangians consist of (sums of)  $(N + 1)$ -*point functions*. They can be regarded as being formed from the  $N$ -fold brackets  $[x_1, x_2, \dots, x_N]$  of the  $L_\infty$ -algebra by evaluation with a dual field via an inner product. In terms of the  $N$ -fold bracket, we then define

$$\{y_0 y_1 \dots y_N\} = \langle y_0 | [y_1, y_2, \dots, y_N] \rangle .$$

Zwiebach presents a classical action in closed string field theory, gauge transformations and shows the invariance of the action. The

classical string action is simply given by

$$S(\Psi) = \frac{1}{2} \langle \Psi, Q\Psi \rangle + \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!} \{\Psi \dots \Psi\}.$$

The gauge transformations of the theory are given by

$$\delta_{\Lambda}|\Psi \rangle = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} [\Psi, \dots, \Psi, \Lambda]$$

for  $\Lambda$  in  $\mathfrak{g}$ . Notice that all the terms of higher order are necessary for these to be consistent.

Similarly, in open string field theory, one can define  $N + 1$ -point functions using the structure maps  $m_N$  of an  $A_{\infty}$ -algebra.

A primer on  $L_{\infty}$  theory for physicists is [LS93].

### 6.3 Open-closed string field theory and OCHA

Having considered both  $A_{\infty}$ - and  $L_{\infty}$ -algebras, we come to the combination known as *OCHA* for Open-Closed Homotopy Algebra [KS06a, KS06b]. Inspired by open-closed string field theories [Zwi98], these involve an  $L_{\infty}$ -algebra acting by derivations (up to strong homotopy) on an  $A_{\infty}$ -algebra but have an additional piece of structure corresponding to a closed string opening to an open string. The details are quite complicated in the original papers, but, just as other “ $\infty$ ” algebras can be characterized by a single coderivation on an appropriate dgc coalgebra, the same has been achieved for OCHAs by Hoefel [Hoe06]. Now, returning to Kadeishvili’s work, in a recent paper [KL], he and Lada have exhibited a very small, concrete example, providing one that perhaps can provide a *toy model* for open-closed string field theory.

## 7 Coda

It has been a pleasure to sketch the connections between Tornike’s work and my own, as well as that of many other contributors to higher homotopy algebra. Surely some further ‘twist’ to this history lies ahead.

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